# Truthful Allocation Mechanisms Without Payments: Characterization and Implications on Fairness

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We study the mechanism design problem of allocating a set of indivisible items without monetary transfers. Despite the vast literature on this very standard model, it still remains unclear how do truthful mechanisms look like. We focus on the case of two players with additive valuation functions and our purpose is twofold. First, our main result provides a complete characterization of truthful mechanisms that allocate all the items to the players. Our characterization reveals an interesting structure underlying all truthful mechanisms, showing that they can be decomposed into two components: a *selection part* where players pick their best subset among prespecified choices determined by the mechanism, and an *exchange part* where players are offered the chance to exchange certain subsets if it is favorable to do so. In the remaining paper, we apply our main result and derive several consequences on the design of mechanisms with fairness guarantees. We consider various notions of fairness, (indicatively, maximin share guarantees and envy-freeness up to one item) and provide tight bounds for their approximability. Our work settles some of the open problems in this agenda, and we conclude by discussing possible extensions to more players.

Additional Key Words and Phrases: mechanism design without money; fair division of indivisible items; maximin share fairness; envy-freeness up to one item

## 1 INTRODUCTION

We study a very elementary and fundamental model for allocating indivisible goods from a mechanism design viewpoint. Namely, we consider a set of indivisible items that need to be allocated to a set of players. An outcome of the problem is an allocation of all the items to the players, i.e., a partition into bundles, and each player evaluates an allocation by his own additive valuation function. Our primary motivation originates from the fair division literature, where such models have been considered extensively. However, the same setting also appears in several domains, including job scheduling, load balancing and many other resource allocation problems.

The focus of our work is on understanding the interplay between truthfulness and fairness in this setting. Hence, we want to identify the effects on fairness guarantees, imposed by eliminating any incentives for the players to misreport their valuation functions. This type of questions has been posed already in previous works and for various notions of fairness, such as envy-freeness, or for the concept of maximin shares (see, among others, Amanatidis et al. 2016; Caragiannis et al. 2009; Lipton et al. 2004). However, the results so far have been rather scarce in the sense that a) in most


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cases, they concern impossibility results which are far from being tight and b) the proof techniques are based on constructing specific families of instances that do not enhance our understanding on the structure of truthful mechanisms, with the exception of Caragiannis et al. (2009) which, however, is only for two players and two items.

In order to comprehend the trade-offs that are inherent between incentives and fairness, we first take a step back and focus solely on truthfulness itself. As is quite common in fair division models, we will not allow any monetary transfers, so that a mechanism simply outputs an allocation of the items. Hence, the question we want to begin with is: *what is the structure of truthful allocation mechanisms?*

There has been already a significant volume of works on characterizing truthful allocation mechanisms for indivisible items, yet there are some important differences from our approach. First, a typical line of work studies this question under the additional assumption of Pareto efficiency or related notions (Ehlers and Klaus 2003; Klaus and Miyagawa 2002; Pápai 2000). The characterization results that have been obtained show that the combination of truthfulness together with Pareto efficiency tends to make the class of available deterministic mechanisms very poor; only some types of dictatorship survive when imposing both criteria. Second, in some cases the analysis is carried out without any restrictions on the class of valuation functions, which again often results in a very limited class of mechanisms (see, e.g., Pápai 2001). When moving to a specific class, such as the class of additive functions which is usualy assumed in fair division, it is conceivable that we can have a much richer class of truthful mechanisms. The results above indicate that the known characterizations of truthful mechanisms are also dependent on further assumptions, which may be well justified in various scenarios, but they are not aligned with the goal of fair division.

## 1.1 Contribution

Our main result is a characterization of deterministic truthful mechanisms that allocate all the items to two players with additive valuations. In doing so, we identify some important allocation properties that every truthful mechanism should satisfy. One such crucial property is the notion of *controlling* items (Definition 3.10); we say that a player controls an item, whenever it is possible to report values that will guarantee him this item, regardless of the other player's valuation function. We show that truthfulness implies that every item is controlled by some player. Exploiting this property further, greatly helps us in understanding how a mechanism operates. Consequently, our analysis and the characterization we eventually obtain reveals an interesting structure underlying all truthful mechanisms; they can all be essentially decomposed into two components: (i) a *selection part* where players pick their best subset among prespecified choices determined by the mechanism, and (ii) an *exchange part* where players are offered the chance to exchange certain subsets if it is favorable to do so. Hence, we call them *picking-exchange mechanisms*.

Next, we apply our main result and derive several consequences on the design of mechanisms with (approximate) fairness guarantees. We consider various notions of fairness in Section 4, starting our discussion with the more standard ones such as proportionality and envy-freeness, and explaining why such concepts cannot be attained—even approximately—by truthful mechanisms. We then focus on more recently studied relaxations of either envy-freeness or proportionality where positive algorithmic results have been obtained (e.g., finding allocations that are envy-free up to one item, or achieve approximate maximin share guarantees). For these notions, we provide tight bounds on the approximation guarantees of truthful mechanisms, settling some of the open problems in this area (Amanatidis et al. 2016; Caragiannis et al. 2009). Interestingly, our results also reveal that the best truthful approximation algorithms for fair division are achieved by *ordinal* mechanisms, i.e., mechanisms that exploit only the relative ranking of the items and not the cardinal information of the valuation functions.

The heart of our approach for obtaining lower bounds on the approximability of fairness criteria, is a necessary condition for fairness in view of our notion of control, which we call *no control of pairs*. It states that no player should control more than one item. We show how this condition summarizes minimum requirements for various fairness concepts previously studied in the literature. Although this condition does not offer an alternative fairness criterion, it is a useful tool for showing lower bounds.

Finally, in Section 5 we provide a general class of truthful mechanisms for the case of multiple players. This class generalizes picking-exchange mechanisms in a non-trivial way. As indicated by our mechanisms, there is a much richer structure in the case of multiple players. In particular, the notion of control does not convey enough information anymore. Instead, there seem to exist several different levels of control.

### 1.2 Related Work

The only work we are aware of, in which a full characterization is given for truthful mechanisms with indivisible items, additive valuations, and no further assumptions is by Caragiannis et al. (2009). However, this is only a characterization for two players and two items. Apart from characterizations, there have been several works that try to quantify the effects of truthfulness on several concepts of fairness. For the performance of truthful mechanisms with respect to envy-freeness, see Caragiannis et al. (2009) and Lipton et al. (2004), whereas for max-min fairness see Bezakova and Dani (2005). Coming to more recent results and along the same spirit, Amanatidis et al. (2016) and Markakis and Psomas (2011) study the notion of maximin share allocations, and a related notion of worst-case guarantees respectively. They obtain separation results, showing that the approximation factors achievable by truthful mechanisms are strictly worse than the known algorithmic (nontruthful) results. Obtaining a better understanding for the structure of truthful mechanisms and how they affect fairness has been an open problem underlying all the above works. For a better and more complete elaboration on fairness and the numerous fairness concepts that have been suggested, we refer the reader to the books (Brams and Taylor 1996; Moulin 2003; Robertson and Webb 1998) and the recent surveys (Bouveret et al. 2016; Procaccia 2016).

There has been a long series of works on characterizing mechanisms with indivisible items beyond the context of fair divison. Many works characterize the allocation mechanisms that arise when we combine truthfulness with Pareto efficiency (see, e.g., Ehlers and Klaus 2003; Klaus and Miyagawa 2002; Pápai 2000). Typically, such mechanisms tend to be dictatorial, and it is also well known that economic efficiency is mostly incompatible with fairness (see, e.g., Bouveret et al. 2016). Another assumption that has been used is nonbossiness, which means that one cannot change the outcome without affecting his own bundle. For instance, Svensson (1999) assumes nonbossiness in a setting where each player is interested in acquiring only one item. For general valuations, this also leads to dictatorial algorithms (Pápai 2001). In most of these works ties are ignored by considering strict preference orders over all subsets of the items, while in some cases it is also allowed for the mechanism not to allocate all the items.

There have also been relevant works for the setting of divisible goods (see, among others, Chen et al. 2013; Cole et al. 2013). We note that for additive valuation functions, a mechanism for divisible items can be interpreted as a randomized mechanism for indivisible items. This connection is already discussed and explored in Aziz et al. (2016); Guo and Conitzer (2010). In our work, we do not study randomized mechanisms, however it is an interesting question to have characterization results for such settings as well. Along this direction, see Mennle and Seuken (2014) where a relaxed notion of truthfulness is studied.

Related to our work is also the literature on exchange markets. These are models where players are equipped with an initial endowment, e.g., a house or a set of items. For the case where players can have multiple indivisible items as their initial endowment, see Pápai (2003) and (2007). Exchange markets provide an example where the existing characterizations go well beyond dictatorships and are closely related to the exchange component of our mechanisms.

Finally, for settings with payments, the work of Dobzinski and Sundararajan (2008), and independently of Christodoulou et al. (2008), provided a characterization of truthful mechanisms with two players and additive valuations when all items are allocated. However, their characterization does not apply to our setting because they make an additional assumption, namely *decisiveness*. It roughly requires that each player should be able to receive any possible bundle of items, by making an appropriate bid. Their motivation is the characterization of truthful mechanisms with bounded makespan (maximum finishing time) for the scheduling problem, and in their case decisiveness is necessary in order to achieve bounded guarantees. In our case, our motivation is fairness, and decisiveness is a very strong assumption which has the opposite effects of what we need; e.g., assigning the full-bundle to a player is unacceptable in terms of fairness. Finally, Christodoulou and Kovács (2011) give a global characterization of envy-free and truthful mechanisms for settings with payments, when there are multiple players but only two items.

## 2 PRELIMINARIES AND NOTATION

With the exception of Section 5, we consider a setting with two players and a set of $m$ indivisible items, $M = \{1, \dots, m\} = [m]$, to be allocated to the players. We assume that each player $i$ has an additive valuation function $v_i$ over the items, so that for every $S \subseteq M$, $v_i(S) = \sum_{j \in S} v_i(\{j\})$. For $j \in M$, we write $v_{ij}$ instead of $v_i(\{j\})$.

We say that $(S_1, S_2, \dots, S_k)$ is a *partition* of a set $S$, if $\bigcup_{i \in [k]} S_i = S$, and $S_i \cap S_j = \emptyset$ for any $i, j \in [k]$ with $i \neq j$. Note that we do not require that $S_i \neq \emptyset$ for all $i \in [n]$. An allocation of $M$ to the players is a partition in the form $S = (S_1, S_2)$. By $\mathcal{M}$ we denote the set of all allocations of $M$.

The set $\mathcal{V}_m$ of all possible profiles is $\mathbb{R}_+^m \times \mathbb{R}_+^m$, i.e., we assume that $v_{ij} > 0$ for every $i \in \{1, 2\}$ and $j \in M$. For some statements we need the assumption that the players' valuation functions are such that no two sets have the same value. So, let $\mathcal{V}_m^{\neq}$ denote the set of such profiles, i.e.,

$$\mathcal{V}_m^{\neq} = \left\{ (v_1, v_2) \in \mathcal{V}_m \;\middle|\; \forall S, T \subseteq [m] \text{ with } S \neq T, \text{ and } \forall i \in \{1, 2\}, \sum_{j \in S} v_{ij} \neq \sum_{j \in T} v_{ij} \right\}.$$

A *deterministic allocation mechanism with no monetary transfers*, or simply a *mechanism*, for allocating all the items in $M = [m]$, is a mapping $\mathcal{X}$ from $\mathcal{V}_m$ to $\mathcal{M}$. That is, for any profile $\mathbf{v}$, the outcome of the mechanism is $\mathcal{X}(\mathbf{v}) = (X_1(\mathbf{v}), X_2(\mathbf{v})) \in \mathcal{M}$, and $X_i(\mathbf{v})$ denotes the set of items player $i$ receives.

A mechanism $\mathcal{X}$ is *truthful* if for any instance $\mathbf{v} = (v_1, v_2)$, any player $i \in \{1, 2\}$, and any $v_i'$:

$$v_i(X_i(\mathbf{v})) \geq v_i(X_i(v_i', v_{-i})).$$

Since we will repeatedly argue about intersections of $X_i(\mathbf{v})$ with various subsets of $M$, we use $X_i^S(\mathbf{v})$ as a shortcut for $X_i(\mathbf{v}) \cap S$, where $S \subseteq M$.

### 2.1 Fairness concepts[1]

Several notions have emerged throughout the years as to what can be considered a fair allocation. We define below the concepts that we will examine in Section 4. Although all concepts can be clearly defined for any number of players, we provide the definitions for two players, since this is the focus of the paper.

We start with two of the most dominant solution concepts in fair division, namely proportionality and envy-freeness.

*Definition 2.1.* An allocation $S = (S_1, S_2)$ is

(1) *proportional*, if $v_i(S_i) \geq \frac{1}{2} v_i(M)$, for $i \in \{1, 2\}$.
(2) *envy-free*, if $v_1(S_1) \geq v_1(S_2)$, and $v_2(S_2) \geq v_2(S_1)$ .

Proportionality was considered in the very first work on fair division by Steinhaus (1948). Envy-freeness was suggested later by Gamow and Stern (1958), and with a more formal argumentation by Foley (1967) and Varian (1974).

Envy-freeness is a stricter notion than proportionality, but even for the latter existence cannot be guaranteed under indivisible goods. One can also consider approximation versions of these problems as follows: Given an instance $I$, let $E(I)$ be the minimum possible envy that can be achieved at $I$, among all possible allocations. We say that a mechanism achieves a $\rho$-approximation, if for every instance $I$, it produces an allocation where the envy between any pair of players is at most $\rho E(I)$. Similarly for proportionality, suppose that an instance $I$ admits an allocation where every player receives a value of at least $\frac{c(I)}{2} v_i(M)$ for some $c(I) \leq 1$. Then a $\rho$-approximation would mean that each player is guaranteed a bundle with value at least $\frac{\rho c(I)}{2} v_i(M)$.

Apart from the approximation versions, the fact that we cannot always have proportional or envy-free allocations gives rise to relaxations of these definitions, with the hope of obtaining more positive results. We describe below three such relaxations, all of which admit either exact or constant-factor approximation algorithms (not necessarily truthful) in polynomial time.

The first such relaxation is the concept of envy-freeness up to one item, where each person may envy another player by an amount which does not exceed the value of a single item in the other player's bundle. Formally:

*Definition 2.2.* An allocation $S = (S_1, S_2)$ is *envy-free up to one item*, if there exists an item $a_1 \in S_1$, and an item $a_2 \in S_2$, such that $v_i(S_i) \geq v_i(S_j \setminus \{a_j\})$, for $i, j \in \{1, 2\}$.

It is quite easy to achieve envy-freeness up to one item, e.g., a round-robin algorithm that alternates between the players and gives them in each step their best remaining item suffices. Other algorithms are also known to satisfy this criterion (see Lipton et al. 2004).

A more interesting relaxation from an algorithmic point of view, comes from the notion of maximin share guarantees, recently proposed by Budish (2011). For two players, the maximin share of a player $i$ is the value that he could achieve by being the cutter in a discretized form of the cut and choose protocol. This is a guarantee for player $i$, if he would partition the items into two bundles so as to maximize the value of the least valued bundle. We define below the approximate version of this notion.

*Definition 2.3.* Given a set of items $[m]$, the *maximin share* of a player $i \in \{1, 2\}$, is

$$\boldsymbol{\mu}_i = \max_{S \in \mathcal{M}} \min\{v_i(S_1), v_i(S_2)\} .$$

[1]The material of this subsection is needed in the sequel only within Section 4.

For $\rho \leq 1$, an allocation $S = (S_1, S_2)$ is called a $\rho$-approximate maximin share allocation if $v_i(S_i) \geq \rho \cdot \boldsymbol{\mu}_i$ , for $i \in \{1, 2\}$.

For two players maximin share allocations always exist and even though they are NP-hard to compute, we have a PTAS by reducing this to standard job scheduling problems. Hence each player can receive a value of at least $(1 - \epsilon)\boldsymbol{\mu}_i$. For a higher number of players, constant factor approximation algorithms also exist (see Amanatidis et al. 2015; Procaccia and Wang 2014).

Finally, a related approach was undertaken by Hill (1987). This work examined what is the worst case guarantee that a player can have as a function of the total number of players and the maximum value of an item across all players. For two players, the following function was identified precisely as the guarantee that can be given to each player. Note that the total value of the items is normalized to 1 in this case.

*Definition 2.4.* Let $V_2 : [0, 1] \to [0, 1/2]$ be the unique nonincreasing function satisfying $V_2(\alpha) = 1/2$ for $\alpha = 0$, whereas for $\alpha > 0$:

$$V_2(\alpha) = \begin{cases} 1 - k\alpha & \text{if } \alpha \in I(2,k) \\ 1 - \frac{(k+1)}{2(k+1)-1} & \text{if } \alpha \in NI(2,k) \end{cases}$$

where for any integer $k \geq 1$, $I(2,k) = \left[\frac{k+1}{k(2(k+1)-1)}, \frac{1}{2k-1}\right]$ and $NI(2,k) = \left(\frac{1}{2(k+1)-1}, \frac{k+1}{k(2(k+1)-1)}\right)$.

Markakis and Psomas (2011) proved that for two players, there always exists an allocation such that each player $i$ receives at least $V_2(\alpha_i)$, where $\alpha_i = \max_{j \in [m]} v_{ij}$. The approximation version of this notion would be to construct allocations where each player receives a value of at least $\rho V_2(\alpha_i)$. Recently, a stricter variant of this guarantee has been provided by Gourvès et al. (2015) (also see Remark 4.9).

## 3 CHARACTERIZATION OF TRUTHFUL MECHANISMS

We present our main characterization result in this section. We start in subsection 3.1 with the main definitions and illustrating examples, and then we state our result in subsection 3.2 along with a road map of the proof. To avoid repetition, when referring to a truthful mechanism $\mathcal{X}$, we mean a truthful mechanism for allocating all the items in $M$ to two players with additive valuation functions.

### 3.1 A Non-Dictatorial Class of Mechanisms

The main result of this section is that every truthful mechanism is a picking-exchange mechanism (Theorem 3.9). Before we make a precise statement, we formally define the types of mechanisms involved and provide illustrating examples.

*Picking Mechanisms.* We start with a family of mechanisms where players make a selection out of choices that the mechanism offers to them. Given a subset $S$ of items, we define a *set of offers* $\mathcal{O}$ on $S$, as a nonempty collection of proper subsets of $S$ that exactly covers $S$ (i.e., $\bigcup_{T \in \mathcal{O}} T = S$), and in which there is no common element that appears in all subsets (i.e., $\bigcap_{T \in \mathcal{O}} T = \emptyset$).

*Definition 3.1.* A mechanism $\mathcal{X}$ is a *picking mechanism*[2] if there exists a partition $(N_1, N_2)$ of $M$, and sets of offers $\mathcal{O}_1$ and $\mathcal{O}_2$ on $N_1$ and $N_2$ respectively, such that for every profile $\mathbf{v}$,

$$X_i(\mathbf{v}) \cap N_i \in \arg\max_{S \in \mathcal{O}_i} v_i(S).$$

[2]Picking mechanisms are a generalization of *truthful* picking sequences for two players (see (Bouveret and Lang 2014)).

Technical nuances aside, such a mechanism can be implemented by first letting player 1 choose his best offer from $O_1$ and giving what remains from $N_1$ to player 2. Then it lets player 2 choose his best offer from $O_2$ and gives what remains from $N_2$ to player 1. The following example illustrates a picking mechanism.

*Example 3.2.* Consider the following mechanism $\mathcal{X}$ on a set $M = \{1, \ldots 6\}$, which first partitions $M$ into $N_1 = \{1,2,3,4\}, N_2 = \{5,6\}$ and then constructs the offer sets $O_1 = \{\{1,2\},\{2,3\},\{4\}\}, O_2 = \{\{5\},\{6\}\}$. On input $\mathbf{v}$, $\mathcal{X}$ first gives to player 1 his best set—with respect to $v_1$—among $\{1,2\}$, $\{2,3\}$ and $\{4\}$, and then gives what remains from $N_1$ to player 2. Next, $\mathcal{X}$ gives to player 2 his best set—according to $v_2$—among $\{5\}$ and $\{6\}$, and then gives what remains from $N_2$ to player 1. $\mathcal{X}$ resolves ties lexicographically, e.g., in case of a tie, $\{1,2\}$ is preferred to $\{4\}$.

It is not hard to see that $\mathcal{X}$ is truthful. For the following input $v$, the circles denote the allocation.

$$v = \begin{pmatrix} 3 & 5 & 5 & 10 & 4 & 2 \\ 2 & 3 & 6 & 1 & 5 & 3 \end{pmatrix}.$$

*Exchange Mechanisms.* We now move to a quite different class of mechanisms. Let $X, Y$ be two disjoint subsets of $M$. We call the ordered pair $(X, Y)$ an exchange deal. Moreover, we say that an exchange deal $(X, Y)$ is *favorable with respect to* $\mathbf{v}$ if $v_1(Y) > v_1(X)$ and $v_2(Y) < v_2(X)$, while it is *unfavorable with respect to* $\mathbf{v}$ if $v_1(Y) < v_1(X)$ or $v_2(Y) > v_2(X)$. Let $S$ and $T$ be two disjoint subsets of items and let $S_1, S_2, \ldots, S_k$ and $T_1, \ldots, T_k$ be two collections of nonempty and pairwise disjoint subsets of $S$ and $T$ respectively. We say then that the set of exchange deals $D = \{(S_1, T_1), (S_2, T_2), \ldots, (S_k, T_k)\}$ on $(S, T)$ is *valid*.

*Definition 3.3.* A mechanism $\mathcal{X}$ is an *exchange mechanism*[3] if there exists a partition $(E_1, E_2)$ of $M$, and a valid set of exchange deals $D = \{(S_1, T_1), \ldots, (S_k, T_k)\}$ on $(E_1, E_2)$, such that for every profile $\mathbf{v}$, there exists a set of indices $I = I(\mathbf{v}) \subseteq [k]$ for which

$$X_1(\mathbf{v}) = \left(E_1 \setminus \bigcup_{i \in I} S_i\right) \cup \bigcup_{i \in I} T_i\,, \quad X_2(\mathbf{v}) = M \setminus X_1\,.$$

Moreover, $I$ contains the indices of every favorable exchange deal with respect to $\mathbf{v}$, but no indices of unfavorable exchange deals.

On a high level, an exchange mechanism initially partitions the items into endowments for the players, and then examines a list of possible exchange deals. Every exchange that improves both players is performed, while every exchange that reduces the value of even one player is avoided. The mechanism may also perform other exchanges where one player is indifferent and the other player can be either indifferent or improved. Whether such exchange deals are materialized or not is up to the tie-breaking rule employed by the mechanism. The following example illustrates an exchange mechanism.

*Example 3.4.* Let $M = \{1, \ldots 5\}$, and consider the following mechanism $\mathcal{Y}$, with $E_1 = \{1,2,3\}$, $E_2 = \{4,5\}$, and a valid set of exchange deals $D = \{(\{2,3\},\{4\})\}$ on $(E_1, E_2)$: One can think of such a mechanism as if $\mathcal{Y}$ initially reserves the set $E_1$ for player 1 and the set $E_2$ for player 2. Then it examines whether exchanging $\{2,3\}$ with $\{4\}$ strictly improves both players, and performs the exchange only if the answer is yes. Mechanism $\mathcal{Y}$ is an example of an *exchange mechanism* with only one possible *exchange deal*. Again, one can see that no player has an incentive to lie.

[3] If we think about $E_1$, $E_2$ as fixed a priori, then exchange mechanisms are a generalization of fixed deal exchange rules in general exchange markets for two players, see (Pápai 2007).

For the following input $v$, the circles denote the allocation produced.

$$v = \begin{pmatrix} 6 & 2 & 3 & 7 & 1 \\ 1 & 6 & 1 & 4 & 7 \end{pmatrix}.$$

*Picking-Exchange Mechanisms.* Finally, we define the class of picking-exchange mechanisms which is a generalization of both picking and exchange mechanisms.

*Definition 3.5.* A mechanism $\mathcal{X}$ is a *picking-exchange mechanism* if there exists a partition $(N_1, N_2, E_1, E_2)$ of $M$, sets of offers $O_1$ and $O_2$ on $N_1$ and $N_2$ respectively, and a valid set of exchange deals $D = \{(S_1, T_1), \ldots, (S_k, T_k)\}$ on $(E_1, E_2)$, such that for every profile $\mathbf{v}$, $X_i(\mathbf{v}) \cap N_i \in \arg\max_{S \in O_i} v_i(S)$ and $X_1(\mathbf{v}) \cap (E_1 \cup E_2) = (E_1 \setminus \bigcup_{i \in I} S_i) \cup \bigcup_{i \in I} T_i$, where $I = I(\mathbf{v}) \subseteq [k]$ contains the indices of all favorable exchange deals, but no indices of unfavorable exchange deals.

It is helpful to think that a picking-exchange mechanism runs independently a picking mechanism on $N_1 \cup N_2$ and an exchange mechanism on $E_1 \cup E_2$, like in Example 3.6. Although this is true under the assumption that the players' valuation functions are such that no two sets have the same value, it is not true for general additive valuations. The reason is that the tie-breaking for choosing the offers from $O_1$ and $O_2$ may not be independent from the decision of whether to perform each exchange that is neither favorable nor unfavorable.

The following example illustrates a picking exchange mechanism.

*Example 3.6.* Let $M = \{1, \ldots, 11\}$, and consider the mechanism $\mathcal{Z}$ that partitions $M$ into $N_1 = \{1,2,3,4\}, N_2 = \{5,6\}$, $E_1 = \{7,8,9\}$ and $E_2 = \{10,11\}$, and is the combination of $\mathcal{X}$ and $\mathcal{Y}$ from the previous two examples: On input $\mathbf{v}$, $\mathcal{Z}$ runs $\mathcal{X}$ on $N_1 \cup N_2$ and $\mathcal{Y}$ on $E_1 \cup E_2$. It outputs the union of the outputs of $\mathcal{X}$ and $\mathcal{Y}$.

For the following input $v$, the circles denote the final allocation.

$$v = \begin{pmatrix} 3 & 5 & 5 & 10 & 4 & 2 & 6 & 2 & 3 & 7 & 1 \\ 2 & 3 & 6 & 1 & 5 & 3 & 1 & 6 & 1 & 4 & 7 \end{pmatrix}.$$

## 3.2 Truthfulness and Picking-Exchange Mechanisms

Essentially, we show that a mechanism is truthful if and only if it is a picking-exchange mechanism. We begin with the easier part of our characterization, namely that under the assumption that each valuation function induces a strict preference relation over all possible subsets, every picking-exchange mechanism is truthful. Recall that the set of such profiles is denoted by $\mathcal{V}_m^{\neq}$.

Theorem 3.7. *When restricted to $\mathcal{V}_m^{\neq}$, every picking-exchange mechanism $\mathcal{X}$ for allocating $m$ items is truthful.*

*Remark 3.8.* For simplicity, Theorem 3.7 is stated for a subclass of additive valuation functions. However, it holds for general additive valuations as long as the mechanism uses a sensible tie-breaking rule (e.g., label-based or welfare-based).[4]

We are now ready to state the main result of this work.

Theorem 3.9. *Every truthful mechanism $\mathcal{X}$ can be implemented as a picking-exchange mechanism.*

The rest of this subsection is a road map to the proof of Theorem 3.9. The proof is long and technical, so for the sake of presentation, it is broken down to several lemmata. In order to illustrate the high-level ideas, the proofs of those lemmata are deferred to the full version of the paper.

[4]Describing all such tie-breaking rules seems to be an interesting, nontrivial question for future work, but not our main focus here.

For the rest of this subsection we assume a truthful mechanism $\mathcal{X}$ for allocating all the items in $M = [m]$ to two players with additive valuation functions. Every statement is going to be with respect to this $\mathcal{X}$.

*3.2.1 The Crucial Notion of Control.* We begin by introducing the notions of *strong desire* and of *control*, which are of key importance for our characterization. We say that player $i$ *strongly desires* a set $S$ if each item in $S$ has more value for him than all the items of $M \setminus S$ combined, i.e., if for every $x \in S$ we have $v_{ix} > \sum_{y \in M \setminus S} v_{iy}$.

*Definition 3.10.* We say that player $i$ *controls* a set $S$ with respect to $\mathcal{X}$, if every time he strongly desires $S$ he gets it whole, i.e., for every $\mathbf{v} = (v_1, v_2)$ in which player $i$ strongly desires $S$, then we have that $S \subseteq X_i(\mathbf{v})$ .

Clearly, given $\mathcal{X}$, any set $S$ can be controlled by at most one player.

The following is a key lemma for understanding how truthful mechanisms operate. The lemma together with Corollary 3.12 below show that every item is controlled by some player under any truthful mechanism.

Lemma 3.11 (Control Lemma). *Let $S \subseteq M$. If there exists a profile $\mathbf{v} = (v_1, v_2)$ such that both players strongly desire $S$, and $S \subseteq X_i(\mathbf{v})$ for some $i \in \{1, 2\}$, then player $i$ controls every $T \subseteq S$ with respect to $\mathcal{X}$.*

Proof. Let $\mathbf{v} = (v_1, v_2)$ be a profile such that both players strongly desire $S$ and $S \subseteq X_1(\mathbf{v})$ (the case where $S \subseteq X_2(\mathbf{v})$ is symmetric). We first prove the statement for $T = S$. Let $\mathbf{v}' = (v_1', v_2')$ be any profile in which player 1 strongly desires $S$, i.e., $v_{1x}' > \sum_{y \in M \setminus S} v_{1y}', \forall x \in S$. Initially, consider the intermediate profile $\mathbf{v}^* = (v_1, v_2')$. If $S \cap X_2(\mathbf{v}^*) \neq \emptyset$ then player 2 would deviate from profile $\mathbf{v}$ to $\mathbf{v}^*$ in order to strictly improve his total utility. So by truthfulness we derive that $S \subseteq X_1(\mathbf{v}^*)$. Similarly, in the profile $\mathbf{v}'$, if $S \cap X_2(\mathbf{v}') \neq \emptyset$ then player 1 would deviate from $\mathbf{v}'$ to $\mathbf{v}^*$ in order to strictly improve. Thus by truthfulness we have $S \subseteq X_1(\mathbf{v}')$. We conclude that player 1 controls $S$.

Now, suppose that $\mathbf{v}'' = (v_1'', v_2'')$ is any profile in which player 1 strongly desires $T \subsetneq S$. If $T \nsubseteq X_1(\mathbf{v}'')$ then player 1 could strictly improve his utility by playing $v_1'$ from before (i.e., he declares that he strongly desires $S$) and getting $S \supsetneq T$. Thus, by truthfulness, $T \subseteq X_1(\mathbf{v}'')$, and we conclude that player 1 controls $T$. □

Notice here that the existence of sets that are controlled by some player is always guaranteed. Specifically, each singleton $\{x\}$ is always controlled (only) by one of the players. Indeed, when both players strongly desire $\{x\}$, it is always the case that $\{x\} \subseteq X_i(\mathbf{v})$ for some $i \in \{1, 2\}$. This is summarized in the following corollary.

Corollary 3.12. *Let $\mathcal{X}$ be a truthful mechanism for allocating the items in $M$ to two players with additive valuations. For every $x \in M$ there exists $i \in \{1, 2\}$ such that only player $i$ controls $\{x\}$ with respect to $\mathcal{X}$.*

Aside from its use in the current proof, the corollary has implications on fairness, that will be explored in Section 4.

*3.2.2 Identifying the Components of a Mechanism.* Our goal now is to determine the "exchange component" and the "picking component" of mechanism $\mathcal{X}$. Every picking-exchange mechanism is completely determined by the seven sets $N_1$, $N_2$, $O_1$, $O_2$, $E_1$, $E_2$, and $D$ mentioned in Definition 3.5 (plus a deterministic tie-breaking rule). Below we try to identify these sets. Later we show that the mechanism's behavior is identical to that of a picking-exchange mechanism defined by them.

To proceed, we will need to consider the collection of all maximal sets controlled by each player. For $i \in \{1,2\}$, let

$$\mathcal{A}_i = \{S \subseteq M \mid \text{player } i \text{ controls } S \text{ and for any } T \supsetneq S,\ i \text{ does not control } T\}\,.$$

Clearly, every set controlled by player $i$ is a subset of an element of $\mathcal{A}_i$. According to Lemma 3.11, if we consider the set $C_i = \bigcup_{S \in \mathcal{A}_i} S$, i.e., the union of all the sets in $\mathcal{A}_i$, this is exactly the set of items that are controlled—as singletons—by player $i$.

Corollary 3.13. *The sets $C_1$ and $C_2$ define a partition of $M$.*

Using the $\mathcal{A}_i$s and the $C_i$s, we define the sets of interest that determine the mechanism. We begin with $E_i = \bigcap_{S \in \mathcal{A}_i} S$ for $i \in \{1,2\}$. As we are going to see eventually in Lemma 3.21, the "exchange component" of $\mathcal{X}$ is observed on $E_1 \cup E_2$.

Defining the corresponding valid set of exchange deals $D$ is trickier, and we need some terminology. Recall that $X_i^S(\mathbf{v}) = X_i(\mathbf{v}) \cap S$. For $S \subseteq E_1$ and $T \subseteq E_2$, we say that $(S,T)$ is a *feasible exchange*, if there exists a profile $\mathbf{v}$, such that $X_1^{E_1 \cup E_2}(\mathbf{v}) = (E_1 \setminus S) \cup T$. In such a case, each of $S$ and $T$ is called *exchangeable*. An exchangeable set $S$ is called *minimally exchangeable* if any $S' \subsetneq S$ is not exchangeable. Finally, a feasible exchange $(S,T)$ is a *minimal feasible exchange*, if at least one of $S$ and $T$ is minimally exchangeable. Now let

$$D = \{(S,T) \mid (S,T) \text{ is a minimal feasible exchange with respect to } \mathcal{X}\}\,.$$

Of course, at this point it is not clear whether $D$ is well defined as a valid set of exchange deals, and this is probably the most challenging part of the characterization.

Next, we define $N_i = C_i \setminus E_i$ and $\mathcal{O}_i = \{S \setminus E_i \mid S \in \mathcal{A}_i\}$ for $i \in \{1,2\}$. As shown in Lemmata 3.14 and 3.15, we identify the "picking component" of $\mathcal{X}$ on $N_1 \cup N_2$, and $\mathcal{O}_i$ will correspond to the set of offers.

Note that by Corollary 3.13 and the above definitions, $(N_1, N_2, E_1, E_2)$ is a partition of $M$. The intuition behind breaking $C_i$ into $N_i$ and $E_i$ is that player $i$ has different levels of control on those two sets. The fact that $E_i$ is contained in every maximal set controlled by player $i$ will turn out to mean that $\mathcal{X}$ gives the ownership of $E_i$ to player $i$. On the other hand, the control of player $i$ on $N_i$ is much more restricted as shown below.

*3.2.3 Cracking the Picking Component.* The first step is to show that the $\mathcal{O}_i$s defined above, greatly restrict the possible allocations of the items of $N_1 \cup N_2$. In particular, whatever player $i$ receives from $N_i$ must be contained in some set of $\mathcal{O}_i$.

Lemma 3.14. *For every profile $\mathbf{v}$ and every $i \in \{1,2\}$, there exists $S \in \mathcal{O}_i$ such that $X_i^{N_i}(\mathbf{v}) \subseteq S$.*

The idea behind the proof of Lemma 3.14 is that by receiving some $X_i^{N_i}(\mathbf{v})$ not contained in any set of $\mathcal{O}_i$, player $i$ is able to extend his control to subsets not contained in $C_i$, thus leading to contradiction. The proof, as many of the proofs of the remaining lemmata, includes the careful construction of a series of profiles, where in each step one has to argue about how the allocation does or does not change.

Given the restriction implied by Lemma 3.14, next we can prove that the subset of $N_i$ that player $i$ receives must be the best possible from his perspective, hence the mechanism behaves as a picking mechanism on each $N_i$. Intuitively, suppose that player 1 receives a subset $S$ of $N_1$ which is not an element of $\mathcal{O}_1$. By Lemma 3.14, $S$ is contained in an element $S'$ of $\mathcal{O}_1$. Since player 1 controls $S'$, this means that he gave up part of his control to gain something that he was not supposed to. Actually, it can be shown that it is the case where player 2 also gave part of his control (either on $N_2$ or $E_2$). This mutual transfer of control, combined with truthfulness, eventually leads to profiles where some of the items must be given to both players at the same time, hence a contradiction.

Lemma 3.15. *For every profile* $\mathbf{v}$ *and every* $i \in \{1,2\}$ *we have* $X_i^{N_i}(\mathbf{v}) \in \arg\max_{S \in O_i} v_i(S)$.

Now we know that $\mathcal{X}$ behaves as the "right" picking-exchange mechanism on $N_1 \cup N_2$. For most of the rest of the proof we would like to somehow ignore this part of $\mathcal{X}$ and focus on $E_1 \cup E_2$.

*3.2.4 Separating the Two Components.* As mentioned right after Definition 3.5, there is some kind of independence between the two components of a picking-exchange mechanism, at least when restricted on $\mathcal{V}_m^{\neq}$. This independence should be present in $\mathcal{X}$ as well; in fact we are going to exploit it to get rid of $N_1 \cup N_2$ until the last part of the proof.

Lemma 3.16. *Let* $\mathbf{v} = (v_1, v_2), \mathbf{v}' = (v_1', v_2') \in \mathcal{V}_m^{\neq}$ *such that* $v_{ij} = v_{ij}'$ *for all* $i \in \{1,2\}$ *and* $j \in E_1 \cup E_2$. *Then* $X_1^{E_1 \cup E_2}(\mathbf{v}) = X_1^{E_1 \cup E_2}(\mathbf{v}')$.

The lemma states that assuming strict preferences over all subsets, the allocation of $E_1 \cup E_2$ does not depend on the values of either player for the items in $N_1 \cup N_2$. What allows this separation is the complete lack of ties in the restricted profile space.

Without loss of generality we may assume that $E_1 \cup E_2 = [\ell]$. We can define a mechanism $\mathcal{X}_E$ for allocating the items of $[\ell]$ to two players with valuation profiles in $\mathcal{V}_\ell^{\neq}$ as

$$\mathcal{X}_E(\mathbf{v}) = (X_1^{E_1 \cup E_2}(\mathbf{v}'), X_2^{E_1 \cup E_2}(\mathbf{v}')), \text{ for every } \mathbf{v} \in \mathcal{V}_\ell^{\neq},$$

where $\mathbf{v}'$ is any profile in $\mathcal{V}_m^{\neq}$ with $v_{ij} = v_{ij}'$ for all $i \in \{1,2\}$ and $j \in [\ell]$. This new mechanism is just the projection of $\mathcal{X}$ on $E_1 \cup E_2$ restricted on a domain where it is well-defined. The truthfulness of $\mathcal{X}_E$ on $\mathcal{V}_\ell^{\neq}$ follows directly from the truthfulness of $\mathcal{X}$ on $\mathcal{V}_m^{\neq}$. Moreover, it is easy to see that player $i$ controls $E_i$ with respect to $\mathcal{X}_E$, for $i \in \{1,2\}$.

The plan is to study $\mathcal{X}_E$ instead of $\mathcal{X}$, show that $\mathcal{X}_E$ is an exchange mechanism, and finally sew the two parts of $\mathcal{X}$ back together and show that everything works properly for any profile in $\mathcal{V}_m$. One issue here is that maybe the set of feasible exchanges with respect to $\mathcal{X}_E$ is greatly reduced, in comparison to the set of feasible exchanges with respect to $\mathcal{X}$, because of the restriction on the domain. In such a case, it will not be possible to argue about exchanges in $D$ that are not feasible anymore. It turns out that this is not the case; the set of possible allocations (of $E_1 \cup E_2$) is the same, whether we consider profiles in $\mathcal{V}_m$ or in $\mathcal{V}_m^{\neq}$.

Lemma 3.17. *For every profile* $\mathbf{v} \in \mathcal{V}_m$ *there exists a profile* $\mathbf{v}' \in \mathcal{V}_m^{\neq}$ *such that* $\mathcal{X}(\mathbf{v}) = \mathcal{X}(\mathbf{v}')$.

In particular, the set of feasible exchanges on $E_1 \cup E_2$ is exactly the same for $\mathcal{X}$ and $\mathcal{X}_E$, and thus we will utilize the following set of exchanges.

$$D = \{(S,T) \mid (S,T) \text{ is a minimal feasible exchange with respect to } \mathcal{X}_E\}.$$

*3.2.5 Cracking the Exchange Component.* In the attempt to show that $\mathcal{X}_E$ is an exchange mechanism, the first step is to show that $D$ is indeed a valid set of exchange deals.

Lemma 3.18. *$D$ is a valid set of exchange deals on* $(E_1, E_2)$.

The above lemma involves three main steps. First we show that each minimally exchangeable set is involved in exactly one exchange deal. Then, we guarantee that minimally exchangeable sets can be exchanged only with minimally exchangeable sets, and finally, we show that minimally exchangeable sets are always disjoint. There is a common underlying idea in the proofs of these steps: whenever there exist two feasible exchanges that overlap in any way, we can construct a profile where both of them are favorable but the two players disagree on which of them is best. On a high level, each player can "block" his least favorable of the conflicting exchanges, and this leads to violation of truthfulness.

Lemma 3.18 implies that every exchangeable set $S \subseteq E_1$ can be decomposed as $S = W \cup \bigcup_{i \in I} S_i$, where $W = S \setminus \bigcup_{i \in I} S_i$ does not contain any minimally exchangeable sets. Ideally, we would like two things. First, the set $W$ in the above decomposition to always be empty, i.e., every exchangeable set should be a union of minimally exchangeable sets. Second, we want every union of minimally exchangeable subsets of $E_1$ to be exchangeable only with the corresponding union of minimally exchangeable subsets of $E_2$, and vice versa. It takes several lemmas and a rather involved induction to prove those. A key ingredient of the inductive step is a carefully constructed argument about the value that each player must gain from any exchange.

Lemma 3.19. *For every exchangeable set $S \subseteq E_1$, there exists some $I \subseteq [k]$ such that $S = \bigcup_{i \in I} S_i$. Moreover, $S$ is exchangeable with $T = \bigcup_{i \in I} T_i$ and only with $T$.*

Finally, we have all the ingredients to fully describe $\mathcal{X}_E$ as an exchange mechanism on $E_1 \cup E_2$ and set of exchange deals $D$.

Lemma 3.20. *Given any profile $\mathbf{v} \in \mathcal{V}_\ell^{\neq}$, each exchange in $D$ is performed if and only if it is favorable, i.e., $X_1^{E_1 \cup E_2}(\mathbf{v}) = (E_1 \setminus \bigcup_{i \in I} S_i) \cup \bigcup_{i \in I} T_i$, where $I \subseteq [k]$ contains exactly the indices of all favorable exchange deals in $D$.*

*3.2.6 Putting the Mechanism Back Together.* As a result of Lemma 3.20 (combined, of course, with Lemmata 3.15 and 3.16), the characterization is complete for truthful mechanisms defined on $\mathcal{V}_m^{\neq}$. For general additive valuation functions, however, we need a little more work. This is to counterbalance the fact that in the presence of ties the allocations of $N_1 \cup N_2$ and $E_1 \cup E_2$ may not be independent.

By Lemmata 3.17 and 3.19, we know that for any $\mathbf{v} \in \mathcal{V}_m$, $X_1^{E_1 \cup E_2}(\mathbf{v})$ is the result of some exchanges of $D$ taking place. There are two things that can go wrong: $\mathcal{X}$ performs an unfavorable exchange, or it does not perform a favorable one. In either of these cases it is possible to construct some profile in $\mathcal{V}_m^{\neq}$ that leads to contradiction. Hence we have the following lemma.

Lemma 3.21. *Given any profile $\mathbf{v} \in \mathcal{V}_m$, $X_1^{E_1 \cup E_2}(\mathbf{v}) = (E_1 \setminus \bigcup_{i \in I} S_i) \cup \bigcup_{i \in I} T_i$, where $I \subseteq [k]$ contains the indices of all favorable exchange deals in $D$, but no indices of unfavorable exchange deals.*

Clearly, Lemma 3.21, together with Lemma 3.15 concludes the proof of Theorem 3.9.

## 3.3 Immediate Implications of Theorem 3.9

As mentioned in Section 1.2, there are several works characterizing truthful mechanisms in combination with other notions, such as Pareto efficiency, nonbossiness, and neutrality (these results are usually for unrestricted, not necessarily additive valuations). Pareto efficiency means that there is no other allocation where one player strictly improves and none of the others are worse-off. Nonbossiness means that a player cannot affect the outcome of the mechanism without changing his own bundle of items. Finally, neutrality refers to a mechanism being consistent with a permutation on the items, i.e., permuting the items results in the corresponding permuted allocation.

Although such notions are not our main focus, the purpose of this short discussion is twofold. On one hand, we illustrate how our characterization immediately implies a characterization for mechanisms that satisfy these extra properties under additive valuations, and on the other hand we see how these properties are either incompatible with fairness or irrelevant in our context.

To begin with, nonbossiness comes for free in our case, since we have two players and all the items must be allocated. Neutrality and Pareto efficiency, however, greatly reduce the space of available mechanisms. Note that it makes more sense to study neutral mechanisms when the valuation functions induce a strict preference order over all sets of items.

Corollary 3.22. *Every neutral, truthful mechanism $\mathcal{X}$ on $\mathcal{V}_m^{\neq}$ can be implemented as a picking-exchange mechanism, such that*

(1) *there exists $i \in \{1,2\}$ such that $E_i = [m]$*, or
(2) *there exists $i \in \{1,2\}$ such that $N_i = [m]$ and $O_i = \{S \subseteq [m] \mid |S| = \kappa\}$ for some $\kappa < m$.*

Corollary 3.23. *Every Pareto efficient, truthful mechanism $\mathcal{X}$ can be implemented as a picking-exchange mechanism, such that*

(1) *there exists $i \in \{1,2\}$ such that $E_i = [m]$*, or
(2) *there exists $j \in [m]$ such that $E_{i_1} = \{j\}$, $E_{i_2} = [m] \setminus \{j\}$, where $\{i_1, i_2\} = \{1,2\}$, and $D = \{(E_1, E_2)\}$*, or
(3) *there exists $i \in \{1,2\}$ such that $N_i = [m]$ and $O_i = \{S \subseteq N_i \mid |S| = m-1\}$.*

It is somewhat surprising that the resulting mechanisms are a strict superset of dictatorships, even when we impose both properties together. Pareto efficiency, however, allows only mechanisms that are rather close to being dictatorial, and thus cannot guarantee fairness of any type. On the other hand, most of the mechanisms defined and studied in Section 4 are neutral, yet neutrality is not implied by the fairness concepts we consider, nor the other way around.

## 4 A NECESSARY FAIRNESS CONDITION AND ITS IMPLICATIONS

In this section, we explore some implications of Theorem 3.9 on fairness properties, i.e., on the design of mechanisms where on top of truthfulness, we would like to achieve fairness guarantees.

In Section 4.1 we show that the Control Lemma implies that truthfulness prevents any bounded approximation for envy-freeness and proportionality. Then, we move on describing a necessary fairness condition, in terms of our notion of "control", that summarizes a common feature of several relaxations of fairness and provide a restricted version of our characterization that follows this fairness condition. This will allow us, in Section 4.2, to examine what this new class of mechanisms can achieve in each of these fairness concepts.

### 4.1 Implications of the Control Lemma.

*4.1.1 Control of singletons.* The basic restriction that truthfulness imposes to every mechanism (leading to poor results for some fairness concepts) comes from Corollary 3.12, an immediate corollary of the Control Lemma, stating that every single item is controlled by some player.

We begin by studing how the above corollary affects two of the most researched notions in the fair division literature, namely *proportionality* and *envy-freeness*. It is well known that even without the requirement for truthfulness, it is impossible to achieve any of these two objectives, simply because in the presence of indivisible goods, envy-free or proportional allocations may not exist. [5]

This leads to the definition of approximation versions of these two concepts for settings with indivisible goods. For example, one could try to construct algorithms such that for every instance, an approximation to the minimum possible envy admitted by the instance is guaranteed. Similarly, approximate proportionality can be considered, i.e., find allocations that achieve an approximation to the best possible value that an instance can guarantee to all agents. See also the discussion in Section 2 on defining the approximation versions of these problems. Note that if time complexity is not an issue, we can always identify the allocation with the best possible envy or with the best possible proportionality, achievable by a given instance.

[5]Consider, for instance, a profile where both players desire only the first item and have a negligible value for the other items. Then one of the players will necessarily remain unsatisfied and receive a value close to zero, no matter what the allocation is.

We are now ready to state our first application, showing that truthfulness prohibits us from having any approximation to the minimum envy or to proportionality. This greatly improves the conclusions of Lipton et al. (2004) and Caragiannis et al. (2009) that truthful mechanisms cannot attain the optimal minimum envy allocation.

Application 4.1. *For any truthful mechanism that allocates all the items to two players with additive valuations, the approximation achieved for either proportionality or the minimum envy is arbitrarily bad (i.e., not lower bounded by any positive function of $m$).*

So far, the conclusion is that even approximate proportionality or envy-freeness are quite stringent and incompatible with truthfulness because of the Control Lemma. The next step would be to relax these notions. There have been already a few approaches on relaxing proportionality and envy-freeness under indivisible goods, leading to solutions such as the maximin share fairness, envy-freeness up to one item (Budish 2011), as well as the type of worst-case guarantees proposed by Hill (1987) (recall Definitions 2.2, 2.3 and 2.4 in Section 2). The fact that a truthful mechanism $\mathcal{X}$ yields control of singletons does not seem to have such detrimental effects on these notions. However, if even a single pair of items is controlled by a player, the same situation arises.

*4.1.2 Control of pairs.* We propose the following *necessary* (but not sufficient) condition that captures a common aspect of all these relaxations of fairness. This allows us to treat all the above concepts of fairness in a unified way.

*Definition 4.2.* We say that a mechanism $\mathcal{X}$ *yields control of pairs* if there exists $i \in \{1, 2\}$ and $S \subseteq [m]$ with $|S| = 2$, such that player $i$ controls $S$ with respect to $\mathcal{X}$.

The following lemma states that in order to obtain impossibility results for the above concepts, it is enough to focus on mechanisms with control of pairs.

Lemma 4.3. *In order to achieve (either exactly or within a bounded approximation) the above mentioned relaxed fairness criteria, a truthful mechanism that allocates all the items to two players with additive valuations cannot yield control of pairs.*

So now we are ready to move to a complete characterization of truthful mechanisms that do not yield control of pairs. Of course such mechanisms are picking-exchange mechanisms, but our fairness condition allows only singleton offers, and the exchange part is completely degenerate.

*Definition 4.4.* A mechanism $\mathcal{X}$ for allocating all the items in $[m]$ to two players is a *singleton picking-exchange mechanism* if it is a picking-exchange mechanism where for each $i \in \{1, 2\}$ at most one of $N_i$ and $E_i$ is nonempty, $|E_i| \leq 1$, and

$$\mathcal{O}_i = \begin{cases} \{\{x\} \mid x \in N_i\} & \text{when } N_i \neq \emptyset \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

i.e., the sets of offers contain all possible singletons.

Hence, typically, in a singleton picking-exchange mechanism player $i$ receives from $N_i \cup E_i$ only his best item. Moreover, for $m \geq 3$, no exchanges are allowed.[6]

Lemma 4.5. *Every truthful mechanism for allocating all the items to two players with additive valuation functions that does not yield control of pairs can be implemented as a singleton picking-exchange mechanism.*

[6]The only exceptions—and the only such mechanisms where both $E_1$ and $E_2$ are nonempty—are two mechanisms for the degenerate case of $m = 2$, e.g., $N_1 = N_2 = \emptyset$, $\mathcal{O}_1 = \mathcal{O}_2 = \{\emptyset\}$, $E_1 = \{a\}$, $E_2 = \{b\}$ and $D = \{(\{a\}, \{b\})\}$, where $\{a, b\} = \{1, 2\}$.

It is interesting to note that, in contrast to Application 4.1, proving Lemma 4.5 without Theorem 3.9 is not straightforward. In fact, it requires a partial characterization which (on a high level) is similar to characterizing the picking component of general mechanisms.

### 4.2 Applications to Relaxed Notions of Fairness

It is now possible to apply Lemma 4.5 on each fairness notion separately, and characterize every truthful mechanism that achieves each criterion.

*Envy-freeness up to one item.* We start with a relaxation of envy-freeness. Below we provide a complete description of the mechanisms that satisfy this criterion.

APPLICATION 4.6. *For $m \le 3$, every singleton picking-exchange mechanism achieves envy-freeness up to one item. For $m = 4$ every singleton picking-exchange mechanism with $|N_1| = |N_2| = 2$ achieves envy-freeness up to one item. Finally, for $m \ge 5$ there is no truthful mechanism that allocates all the items to two players and achieves envy-freeness up to one item.*

*Maximin share fairness and related notions.* For maximin share allocations a truthful mechanism was suggested by Amanatidis et al. (2016) for any number of items and any number of players. For two players, their mechanism is the singleton picking-exchange mechanism with $N_1 = [m]$ and produces an allocation that guarantees to each player a $\frac{1}{\lfloor m/2 \rfloor}$-approximation of his maximin share. It was left as an open problem whether a better truthful approximation exists. Here we show that this approximation is tight; in fact, almost any other singleton picking-exchange mechanism performs strictly worse. Note that the best previously known lower bound for two players was 1/2.

APPLICATION 4.7. *For any $m$ there exists a singleton picking-exchange mechanism that guarantees to player $i$ a $\lfloor \max\{2, m\}/2 \rfloor^{-1}$-approximation of $\boldsymbol{\mu}_i$, for $i \in \{1, 2\}$. There is no truthful mechanism that allocates all the items to two players and achieves a better guarantee with respect to maximin share fairness.*

Regarding now allocations that guarantee an approximation of the function $V_2(\alpha_i)$ defined by Hill (1987) (recall the definition in Section 2), the singleton picking-exchange mechanism with $N_1 = [m]$ was also suggested by Markakis and Psomas (2011) as a $\frac{1}{\lfloor m/2 \rfloor}$-approximation of $V_2(\alpha_i)$.[7] This comes as no surprise, since there exists a strong connection between maximin shares and the function $V_n$, especially for two players. This is illustrated in the following corollary, where both the positive and the negative results coincide with the ones for the maximin share fairness.

APPLICATION 4.8. *For any $m$ there exists a singleton picking-exchange mechanism that guarantees to player $i$ a $\lfloor \max\{2, m\}/2 \rfloor^{-1}$-approximation of $V_2(\alpha_i)$, for $i \in \{1, 2\}$, where $\alpha_i = \max_{j \in [m]} v_{ij}$. There is no truthful mechanism that allocates all the items to two players and achieves a better guarantee with respect to the $V_2(\alpha_i)$s.*

Again, the best previously known lower bound for two players was constant, namely 2/3 due to Markakis and Psomas (2011). In Applications 4.7 and 4.8, it is stated that there exists a $\frac{1}{\lfloor m/2 \rfloor}$-approximate singleton picking-exchange mechanism. It is interesting that *any* singleton picking-exchange mechanism does not perform much worse. Following the corresponding proofs, we have that even the worst singleton picking-exchange mechanism achieves a $\frac{1}{m-1}$-approximation in each case.

*Remark 4.9.* Gourvès et al. (2015) introduced a variant of $V_n$, called $W_n$, and showed that there always exists an allocation such that each player $i$ receives $W_n(\alpha_i) \ge V_n(\alpha_i)$ (where the inequality

[7] The approximation factor in (Markakis and Psomas 2011) is expressed in terms of $V_2(1/m)$, but it simplifies to $\lfloor m/2 \rfloor^{-1}$.

is often strict). Since the definition of $W_n$ is rather involved even for $n = 2$, we defer a formal discussion about it to the full version of the paper. However, it is not hard to show that for every valuation function $v_i$ we have $V_2(\alpha_i) \le W_2(\alpha_i) \le \boldsymbol{\mu}_i$ and thus the analog of Application 4.8 holds.

*Remark 4.10.* Amanatidis et al. (2016) made the following interesting observation: every single known truthful mechanism achieving a bounded approximation of maximin share fairness is *ordinal*, in the sense that it only needs a ranking of the items for each player rather than his whole valuation function. Finding truthful mechanisms that explicitly take into account the players' valuation functions in order to achieve better guarantees was posed as a major open problem. Note that, weird tie-breaking aside, all singleton picking-exchange mechanisms are ordinal! Therefore, from the mechanism designer's perspective, it is impossible to exploit the extra cardinal information given as input and at the same time maintain truthfulness and some nontrivial fairness guarantee.

## 5 TRUTHFUL MECHANISMS FOR MANY PLAYERS

We introduce a family of non-dictatorial, truthful mechanisms for any number of players. Our mechanisms are defined recursively; in analogy to serial dictatorships, the choices of a player define the sub-mechanism used to allocate the items to the remaining players. Here, however, this serial behavior is observed "in parallel" in several sets of a partition of $M$.

A *generalized deal* between $k$ players is a collection of (up to $k(k-1)$) exchange deals between pairs of players. A set $D$ of generalized deals is called *valid* if all the sets involved in all these exchange deals are nonempty and pairwise disjoint. Given a profile $\mathbf{v} = (v_1, v_2, \dots, v_n)$ we say that a generalized deal is *favorable* if it strictly improves all the players involved, while it is *unfavorable* if there exists a player involved whose utility strictly decreases.

*Definition 5.1.* A mechanism $\mathcal{X}$ for allocating all the items in $[m]$ to $n$ players is called a *serial picking-exchange mechanism* if

(1) when $n = 1$, $\mathcal{X}$ always allocates the whole $[m]$ to player 1.
(2) when $n \ge 2$, there exist a partition $(N_1, \dots, N_n, E_1, \dots, E_n)$ of $[m]$, sets of offers $\mathcal{O}_i$ on $N_i$ for $i \in [n]$, a valid set $D$ of generalized deals, and a mapping $f$ from subsets of $M$ to serial picking-exchange mechanisms for $n-1$ players, such that for every profile $\mathbf{v} = (v_1, \dots, v_n)$ we have for all $i \in [n]$:
   - $X_i^{N_i}(\mathbf{v}) \in \arg\max_{S \in \mathcal{O}_i} v_i(S)$,
   - $X_i^{E}(\mathbf{v})$, where $E = \bigcup_{j \in [n]} E_j$, is the result of starting with $E_i$ and performing some of the deals in $D$, including all the favorable deals but no unfavorable ones,
   - the items of $N_i \setminus X_i^{N_i}(\mathbf{v})$ are allocated to players in $[n] \setminus \{i\}$ using the serial picking-exchange mechanism $f\left(N_i \setminus X_i^{N_i}(\mathbf{v})\right)$.

Clearly, serial picking-exchange mechanisms are a generalization of picking-exchange mechanisms studied in Section 3. The following example illustrates how such a mechanism looks like for three players.

*Example 5.2.* Suppose that we have three players with additive valuations. For simplicity, assume that each player's valuation induces a strict preference over all possible subsets of items. Let $M = [100]$ be the set of items, and consider the following relevant ingredients of our mechanism:

- $N_1 = \{1, 2, \dots, 20\}$, $\mathcal{O}_1 = \{\{1, 2, 3\}, N_1 \setminus \{1\}\}$
- $N_2 = \{21, 22, \dots, 50\}$, $\mathcal{O}_2 = \{S \subseteq N_2 \mid |S| = 6\}$
- $N_3 = \{51, 52, \dots, 70\}$, $\mathcal{O}_3 = \{\{51, \dots, 60\}, \{61, \dots, 70\}\}$
- $E_1 = \{71, \dots, 80\}$, $E_2 = \{81, \dots, 90\}$, $E_3 = \{91, \dots, 100\}$
- $D = \left\{ \left[ (\{75, 79\}, \{83\})^{1,3} \right], \left[ (\{71\}, \{88\})^{1,2}, (\{72, 80\}, \{95\})^{1,3}, (\{85\}, \{99, 100\})^{2,3} \right] \right\}$

- $f$ is a mapping from subsets of $M$ to picking-exchange mechanisms (for 2 players)

The above sets are the analog of the corresponding sets of a picking-exchange mechanism. The deals, however, are a bit more complex. For instance, by $\left[(\{71\},\{88\})^{1,2},(\{72,80\},\{95\})^{1,3},(\{85\},\{99,100\})^{2,3}\right]$ we denote the deal in which:

- player 1 gives item 71 to player 2 and items 72, 80 to player 3
- player 2 gives item 88 to player 1 and item 85 to player 3
- player 3 gives item 95 to player 1 and items 99, 100 to player 2

The mapping $f$ suggests which truthful mechanism should be used every time there are items left to be allocated to only two players.

We are ready to describe our mechanism $\mathcal{X}$:

(1) The mechanism gives endowments $E_1, E_2, E_3$ to the three players and then performs each exchange deal that strictly improves all the players involved.
(2) Then, for each $i \in \{1,2,3\}$, the mechanism gives to player $i$ his best set in $O_i$, say $S_i$.
(3) Finally, for each $i \in \{1,2,3\}$, $\mathcal{X}$ uses mechanism $f(N_i \setminus S_i)$ to allocate the items of $N_i \setminus S_i$ to players in $\{1,2,3\} \setminus i$.

Like picking-exchange mechanisms, serial picking-exchange mechanisms are truthful, given an appropriate tie-breaking rule (e.g., a label-based tie-breaking rule). To bypass a general discussion about tie-breaking, however, we may assume that each player's valuation induces a strict preference over all subsets of $M$. We denote by $\mathcal{V}_{n,m}^{\neq}$ the set of profiles that only include such valuation functions. Following almost the same proof, however, we have that for general additive valuations every serial picking-exchange mechanism is truthful when using label-based tie-breaking.

Theorem 5.3. *When restricted to $\mathcal{V}_{n,m}^{\neq}$, every serial picking-exchange mechanism $\mathcal{X}$ for allocating $m$ items to $n$ players is truthful.*

## 6 DISCUSSION

We obtained a nontrivial characterization for truthful mechanisms, that has immediate implications on fairness. A natural question to ask is whether our characterization can be extended for more than two players. Characterizing the truthful mechanisms without money for any number of additive players is, undoubtedly, a fundamental open problem. However, as indicated by Definition 5.1, there seems to be a much richer structure when one attempts to describe such mechanisms, even though serial picking-exchange mechanisms are only a subset of nonbossy truthful mechanisms. In particular, the notion of control that was crucial for identifying the structure of truthful mechanisms for two players does not convey enough information anymore. Instead, there seem to exist several different levels of control, and understanding this structure still remains a very interesting and intriguing question.

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